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Counter-Examples to the Generalized Positive Action Conjecture

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Abstract. We give examples of complete locally asymptotically flat Riemannian 4-manifolds with zero scalar curvature and negative mass. The generalized positive action conjecture of Hawking and Pope [5] is therefore false.

The positive action theorem, proved by Schoen and Yau [9], states that any complete asymptotically flat Riemannian 4-manifold (M, g) with scalar curvature $R=0$ satisfies

$$\lim_{r\to\infty}\int_{S_r}(g_{jk,k}-g_{kk,j})*dS^j \geqq 0$$

with equality iff g is flat; here the integrand is to be computed in an asymptotic coordinate system in which the metric is of the form

$$g_{jk}=\delta_{jk}+0\left(\frac{1}{r^2}\right)$$

and S_r denotes the Euclidean sphere of radius r. For simplicity we will call the left-hand side of the above inequality the *mass* of (M, g), since this expression is the analog of the ADM mass of an asymptotically flat 3-manifold. The *generalized positive action conjecture* of Hawking and Pope [5] asserts that the mass is also non-negative for all *locally asymptotically flat* Riemannian 4-manifolds with scalar curvature $R=0$, and that, moreover, the mass vanishes iff the manifold is Ricci flat with self-dual Weyl curvature.

Unfortunately, as we will see, this plausible-sounding extension does not hold water. We will produce an infinite number of complete locally asymptotically flat Riemannian 4-manifolds with $R=0$ for which the mass is negative. These metrics are Kähler and live on the total spaces of complex line-bundles over $S^2=\mathbb{CP}_1$ for which the first Chern class satisfies $c_1<-2$. They have isometry group $U(2)$, and, by virtue of being Kähler with $R=0$, have anti-self-dual Weyl curvature; cf. [2, 7, 8].

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