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Holomorphic Curves in Loop Groups

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Abstract. It was observed by Atiyah that there is a correspondence between based gauge equivalence classes of SU_n-instantons over S^4 of charge d on the one hand, and based holomorphic curves of genus zero in ΩSU_n of degree d on the other hand. In this paper we study the parameter space of such holomorphic curves which have the additional property that they lie entirely in the subgroup $\Omega_{\mathrm{alg}} SU_n$ of algebraic loops. We describe a cell decomposition of this parameter space, and compute its complex dimension to be $(2n-1)d$.

1. Introduction

It is well known that the space ΩG of (smooth) basepoint preserving maps from the circle S^1 to a compact Lie group G is, in a natural way, a complex manifold. One of its many remarkable properties is that, despite being infinite dimensional, ΩG behaves in many ways as if it were a compact manifold. For example, every holomorphic function $\Omega G \to \mathbb{C}$ is constant. Atiyah [At] proved that, for any compact, complex manifold M, the set of all basepoint preserving holomorphic maps $M \to \Omega G$ lying in a given homotopy class is finite dimensional; in simple cases, the dimension can even be computed. The argument in [At], however, is non-constructive. The purpose of this paper is to complement [At] by giving an explicit geometric construction of a large family of holomorphic maps $M \to \Omega G$ in the case where M is the Riemann sphere S^2. Some examples of where the study of holomorphic maps $M \to \Omega G$ occurs in the literature are given at the end of the introduction.

To describe our results more precisely, we assume, without loss of generality, that G has only one simple factor, so that $\pi_2(\Omega G) \cong \pi_3(G) \cong \mathbb{Z}$. Then any (continuous) map $S^2 \to \Omega G$ has an integer invariant, its degree, given by the induced map on π_2, and this determines the map up to homotopy. Let $\mathrm{Hol}_d^*(S^2, \Omega G)$ denote the set of holomorphic maps $f: S^2 \to \Omega G$ of degree d, which are basepoint preserving in the sense that $f(\infty) = e$, where we think of $S^2 = \mathbb{C} \cup \{\infty\}$ as the extended complex plane and e is the identity element or identity loop in G. Here, d is necessarily $\geqq 0$. Then [At] gives: