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Topological Sigma Models

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Abstract. A variant of the usual supersymmetric nonlinear sigma model is described, governing maps from a Riemann surface Σ to an arbitrary almost complex manifold M. It possesses a fermionic BRST-like symmetry, conserved for arbitrary Σ, and obeying $Q^2 = 0$. In a suitable version, the quantum ground states are the $1 + 1$ dimensional Floer groups. The correlation functions of the BRST-invariant operators are invariants (depending only on the homotopy type of the almost complex structure of M) similar to those that have entered in recent work of Gromov on symplectic geometry. The model can be coupled to dynamical gravitational or gauge fields while preserving the fermionic symmetry; some observations by Atiyah suggest that the latter coupling may be related to the Jones polynomial of knot theory. From the point of view of string theory, the main novelty of this type of sigma model is that the graviton vertex operator is a BRST commutator. Thus, models of this type may correspond to a realization at the level of string theory of an unbroken phase of quantum gravity.

1. Introduction

In recent years, Yang–Mills instantons have played an important role in the study of four manifolds and three manifolds in the work of Donaldson [1] and Floer [2], respectively. More recently, Atiyah advocated an interpretation of Floer theory in terms of a non-relativistic version of supersymmetric quantum Yang–Mills theory [3] and offered some evidence that this might have a relativistic generalization that would account for many features of Donaldson and Floer theory. A relativistic quantum field theory that seems to have the requisite properties was indeed formulated in [4]. It possesses a global fermionic symmetry which is similar in many ways to BRST symmetry, though it can be obtained by twisting ordinary $N = 2$ supersymmetric Yang–Mills theory.

There is also a $1 + 1$ dimensional version of Floer theory [2], which has given striking results about symplectic diffeomorphisms of symplectic manifolds. From

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