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Ergodic Endomorphisms of Compact Abelian Groups*

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Abstract. We show that for a surjective endomorphism of a compact abelian group ergodicity is equivalent to a condition which implies r-mixing for all $r \geqq 1$, and we characterize such maps algebraically. This is then used in proving the ergodicity of an extensive class of endomorphisms of the binary sequence space. As a simple corollary it is found that one-dimensional linear cellular automata and the accumulator automata are r-mixing for all $r \geqq 1$.

1. Introduction

Let θ be a continuous automorphism of a compact abelian group G. The classical automorphism theorem of Halmos [4] states that θ is strongly mixing with respect to the normalized Haar measure on G if and only if it is ergodic. To explain our generalization, say θ is *completely mixing* in G if and only if the following holds:

Given any integer $r \geqq 1$, any $r+1$ measurable subsets $A_0, \dots, A_r$ of G, and any r sequences $\{k_{in}\}$ of positive integers such that $\lim_{n\to\infty} k_{in} = \infty$ for all $1 \leqq i \leqq r$, we have

$$\lim_{n\to\infty} \mu(A_0 \cap \theta^{-k_{1n}}(A_1) \cap \cdots \cap \theta^{-k_{rn}}(A_n)) = \prod_{j=0}^{r} \mu(A_j).$$

In an arbitrary space the above property may well be stronger than the condition of r-mixing introduced by Rohlin [9], since the latter involves the extra assumption that $\lim_{n\to\infty} \min_{i \neq j} |k_{in} - k_{jn}| = \infty$. We prove the following generalization of the above result.

Theorem 1 (Endomorphism Theorem). *Let G be a compact abelian group with normalized Haar measure μ, and let θ be a continuous surjective endomorphism of G. Then θ is μ-invariant. Moreover, the following are equivalent:*

(i) *θ is completely mixing.*
(ii) *θ is r-mixing for all $r \geqq 1$.*

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