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On the Asymptotic Behaviour of the L^2-Norm of Suitable Weak Solutions to the Navier–Stokes Equations in Three-Dimensional Exterior Domains

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Abstract. We prove L^2-decay rates of suitable weak solutions to the Navier–Stokes equations in exterior domains. The results for the order of decay are the same as for the solutions to the Cauchy problem of the Navier–Stokes equations. Finally in the case of $\Omega = R^3$ the decay rate order is sharp in the class of solutions considered by us.

1. Introduction

Recently, the problem of the asymptotic behaviour of the kinetic energy of an incompressible viscous fluid, governed by the Navier–Stokes equations, when the region of motion is unbounded in all directions, has been studied by several authors, cf. [3, 4, 7, 13, 14, 16, 19, 20, 23, 24]. Formally, this question is reduced to the asymptotic behaviour of the L^2-norm of solutions to the Navier–Stokes equations. The results of [3, 4, 7, 13, 14, 16, 19, 20, 23, 24] can be essentially devided in two groups. In [3, 4, 13, 14, 16] the asymptotic behaviour of the L^2-norm of solutions is obtained when the region Ω of motion of the fluid is an exterior domain, while the other works concern the asymptotic behaviour of the L^2-norm of solutions to a Cauchy problem for the Navier–Stokes equations. As regards the case of an exterior domain, in [13] the asymptotic behaviour of the L^2-norm of weak solutions to the Navier–Stokes equations is proved when the weak solutions verify the energy inequality in the "strong" form:

$$|\mathbf{v}(t)|^2 + 2\int_s^t |\nabla \mathbf{v}(\tau)|^2 \, d\tau \leqq |\mathbf{v}(s)|^2 \quad \forall t \geqq s \quad \text{and} \quad \text{a.e.} \quad \text{for} \quad s \geqq 0, \tag{I}$$

($|\cdot|$ is the L^2-norm of solution $\mathbf{v}$). However, relation (I) is not an a priori estimate for weak solutions to the Navier–Stokes equations on exterior domains. This fact makes formal the results obtained in [13] except that in the particular cases of a Cauchy problem and of initial-boundary value problem in exterior domains, where the initial data of the solutions are "small" in a suitable sense (global solution of the type furnished in [5, 9]). Subsequently, in [4] the relation (I) is determined for