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Theory of Matching Rules for the 3-Dimensional Penrose Tilings

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Abstract. We consider packings of the two Ammann rhombohedra used for tiling the three dimensional space. We define decorations for the facets of the rhombohedra. Using elementary algebraic topology, we prove that any tiling by these rhombohedra with matching decorations is a quasiperiodic Penrose tiling. The proof does not involve any reference to self similarity.

1. Introduction

Since their invention by Roger Penrose, his well-known aperiodic and five-fold symmetric tilings of the planes have motivated numerous works [1–3]. See [4] for a review on the Penrose and related tilings. The construction and therefore a possible description of these tilings is controlled by their strong self-similarity properties, known as "inflation" and "deflation", which give access to some of their main properties, and specifically to their aperiodicity. On the other hand, these tilings can be obtained through a local "growth process" constrained by the so-called matching rules: there exists a set of decorations of the edges of the tiles (which can of course be realized in many different ways), such that any infinite tilings in which the decorations of the edges of any adjacent tiles coincide is a Penrose tiling.

Although the Penrose tilings had early attracted the interest of some crystallographers [5] speculating on their possible implications for solid state physics (and who have first empirically observed their quasiperiodicity using optical transforms), it is the discovery of icosahedral quasicrystals [6] in rapidly cooled alloys of Aluminium and Manganese which has triggered an intense interest for these structures in the solid state physics community. See [7] for a general review. Three dimensional analogues of the Penrose tilings were soon devised [8, 9] and their quasiperiodicity proved [9]. These 3-dimensional tilings are packings of two rhombohedra with identical facets known as Ammann rhombohedra, and they are icosahedrally symmetric in the same loose sense as the Penrose tilings are five-fold symmetric.

Up to now, only global methods such as the strip projection method are known to construct these tilings, and it was interesting to look for matching rules