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The Action Functional in Non-Commutative Geometry

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Abstract. We establish the equality between the restriction of the Adler-Manin-Wodzicki residue or non-commutative residue to pseudodifferential operators of order $-n$ on an n-dimensional compact manifold M, with the trace which J. Dixmier constructed on the Macaev ideal. We then use the latter trace to recover the Yang Mills interaction in the context of non-commutative differential geometry.

Introduction

The non-commutative residue was discovered in the special case of one dimensional symbols by Adler [1] and Manin [8] in the context of completely integrable systems. In a quite remarkable work [13], Wodzicki proved that it could still be defined in arbitrary dimension and gave the only non-trivial trace, noted Res, for the algebra of pseudodifferential operators of arbitrary order. Given such an operator P on the manifold M, $\mathrm{Res}\,P$ is the coefficient of $\mathrm{Log}\,t$ in the asymptotic expansion of $\mathrm{Trace}(P\,e^{-t\Delta})$, where Δ is a Laplacian. Equivalently it is the residue at $s=0$ of the ζ function $\zeta(s)=\mathrm{Trace}(P\Delta^{-s})$. It is *not* the usual regularisation $\zeta(0)$ of the trace, and it vanishes on any P of order strictly less than $-\dim M$, and on any differential operator. In general this trace: Res, has no positivity property, i.e. one does not have $\mathrm{Res}(P^*P)\geqq 0$. However its restriction to operators of order $-n$, $n=\dim M$ is positive. This restriction of Res to pseudodifferential operators of order $-n$ was discovered and studied by Guillemin [14]. Even though it is easier to handle than the general residue, it will be of great help for our purpose which is to show how conformal geometry fits with [3], the case of Riemannian geometry being treated in [5].

Our first result is the equality between Res and a trace on the dual Macaev ideal, introduced by Dixmier in [6] in order to show that the von Neumann algebra $\mathscr{L}(\mathscr{H})$ of all bounded operators in Hilbert space possessed non-trivial tracial weights. I am grateful to J. Dixmier for explaining his result to me and to *D.* Voiculescu for helpful conversations on the subject of Macaev ideals. Thus we recall that, given a Hilbert space $\mathscr{H}$, the Macaev ideal $\mathscr{L}^{\omega}(\mathscr{H})$ is the ideal of