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Remarks on Cosmological Spacetimes and Constant Mean Curvature Surfaces

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Abstract. We give a general condition which ensures the existence of constant mean curvature (CMC) Cauchy surfaces in cosmological spacetimes. However, there is an example of a spacetime which does not satisfy this condition and does not admit any CMC Cauchy surfaces. We discuss conditions under which CMC surfaces may exist.

Introduction

A Lorentzian manifold $\mathscr{V}$ is a *cosmological spacetime* if it is globally hyperbolic with compact Cauchy surfaces and satisfies the timelike convergence condition

(TCC) $$\operatorname{Ric}(T, T) \geqq 0, \text{ for every timelike vector } T.$$

In this paper we are concerned with the following conjecture:

Conjecture 1. There is a CMC Cauchy surface in $\mathscr{V}$.

We first show the following existence result:

Theorem. *Suppose the condition*

(G) $$\mathscr{V} - I(p) \text{ is compact, for some point } p \in \mathscr{V},$$

is satisfied. Then there is a regular CMC Cauchy surface, which passes through p.

By *regular surface* we mean a strictly spacelike $C^{2,\alpha}$ hypersurface [B2] and we are assuming throughout that the spacetime metric is C^2. The proof of the theorem depends on some new existence and regularity results for prescribed mean curvature surfaces [B2].

However, Conjecture 1 is false in general, as we have:

Example 1. There is a cosmological spacetime $\mathscr{V}$ which admits no CMC Cauchy surfaces.