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Rationality in Conformal Field Theory

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Abstract. We show that if the one-loop partition function of a modular invariant conformal field theory can be expressed as a finite sum of holomorphically factorized terms then c and all values of h are rational.

1. Introduction

There is some evidence for an underlying arithmetic nature of conformal field theory. If this is so then the fundamental data of the theory might be expected to have an arithmetic significance. These data include the central extension c and highest weights $\{h_i\}$ of representations of the Virasoro algebra, together with operator product expansion coefficients [1]. While the arithmetic nature of these data is still almost wholly conjectural, it is interesting to note that in all known conformal field theories the value of c is rational. In this note we prove that in a certain subclass of conformal theories the values of c (and of h_i) must be rational.

The class of conformal theories we will discuss are known as rational conformal field theories. Conformal field theories are a distinguished class of two-dimensional quantum field theories that are partially characterized by the requirement that the Hilbert space of the theory H be a representation of a product of commuting Virasoro algebras: $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ of the form

$$H = \oplus_{a,b \geqq 0} V(h_a, c) \otimes \bar{V}(\bar{h}_b, c), \tag{1.1}$$

where $V(h, c)$ is the irreducible highest weight representation characterized by the central extension c (the same for all representations in (1.1)) and highest weight h, i.e. $L_0 v = hv$ for the highest weight vector v. In conformal field theory $h_a \geqq 0$, and the degeneracy of the states with $h = \bar{h} = 0$ is exactly one. Another distinguishing characteristic of conformal field theory is modular invariance, which, among other things states that the one loop partition function:

$$Z(\tau) = \mathrm{Tr}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} = \sum_{a,b \geqq 0} N_{ab} \chi(h_a, c) \bar{\chi}(\bar{h}_b, c) \tag{1.2}$$

is modular invariant. Here N_{ab} is the degeneracy of representations $(h_a, \bar{h}_b)$, $\chi(h, c)$ is the character of the representation $V(h, c)$, and $q = e^{2\pi i \tau}$, where $\tau \in \mathscr{H}$, the upper