Commun. Math. Phys. 117, 1–36 (1988)

Communications in
Mathematical
Physics

Moduli Spaces of Curves and Representation Theory

E. Arbarello[1], C. De Concini[2], V. G. Kac[3], and C. Procesi[1]

[1] Dipartimento di Matematica, Universita Degli Studi di Roma “La Sapienza”, Rome, Italy
[2] Dipartimento di Matematica, Universita di Roma II “Tor Vergata”, Rome, Italy
[3] Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

Abstract. We establish a canonical isomorphism between the second cohomology of the Lie algebra of regular differential operators on $\mathbb{C}^\times$ of degree $\leqq 1$, and the second singular cohomology of the moduli space $\hat{\mathscr{F}}_{g-1}$ of quintuples $(C, p, z, L, [\varphi])$, where C is a smooth genus g Riemann surface, p a point on C, z a local parameter at p, L a degree $g-1$ line bundle on C, and $[\varphi]$ a class of local trivializations of L at p which differ by a non-zero factor. The construction uses an interplay between various infinite-dimensional manifolds based on the topological space H of germs of holomorphic functions in a neighborhood of 0 in $\mathbb{C}^\times$ and related topological spaces. The basic tool is a canonical map from $\hat{\mathscr{F}}_{g-1}$ to the infinite-dimensional Grassmannian of subspaces of H, which is the orbit of the subspace H_- of holomorphic functions on $\mathbb{C}^\times$ vanishing at ∞, under the group Aut$H$. As an application, we give a Lie-algebraic proof of the Mumford formula: $\lambda_n = (6n^2 - 6n + 1)\lambda_1$, where λ_n is the determinant line bundle of the vector bundle on the moduli space of curves of genus g, whose fiber over C is the space of differentials of degree n on C.

Introduction

Consider the Lie algebra $\mathscr{D}^F$ (F for finite) of regular differential operators of degree less than or equal to 1 on $\mathbb{C}^\times$ and its subalgebra $\mathbf{d}^F$ of vector fields, so that $\left\{z^j, d_j = z^{j+1}\frac{d}{dz}\right\}_{j\in\mathbb{Z}}$ is a basis of $\mathscr{D}^F$ and $\{d_j\}_{n\in\mathbb{Z}}$ is a basis of $\mathbf{d}^F$. The Lie algebra $\mathscr{D}^F$ acts in a natural way on the space V_n of regular differentials of degree n on $\mathbb{C}^\times$ with basis $v_k = z^{-k}dz^n$, $k \in \mathbb{Z}$. This gives an inclusion

$$\phi_n : \mathscr{D}^F \to \mathbf{a}_\infty^F ,$$

where $\mathbf{a}_\infty^F$ is the Lie algebra of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ such that $a_{ij} = 0$ for $|i-j| \gg 0$. We also consider the restriction of ϕ_n to $\mathbf{d}^F$:

$$\varrho_n : \mathbf{d}^F \to \mathbf{a}_\infty^F .$$