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The Geometry of Super Riemann Surfaces

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Abstract. We define super Riemann surfaces as smooth $2|2$-dimensional supermanifolds equipped with a reduction of their structure group to the group of invertible upper triangular 2×2 complex matrices. The integrability conditions for such a reduction turn out to be (most of) the torsion constraints of 2d supergravity. We show that they are both necessary and sufficient for a frame to admit local superconformal coordinates. The other torsion constraints are merely conditions to fix some of the gauge freedom in this description, or to specify a particular connection on such a manifold, analogous to the Levi–Civita connection in Riemannian geometry. Unlike ordinary Riemann surfaces, a super Riemann surface cannot be regarded as having only one complex dimension. Nevertheless, in certain important aspects super Riemann surfaces behave as nicely as if they had only one dimension. In particular they posses an analog $\hat{\partial}$ of the Cauchy–Riemann operator on ordinary Riemann surfaces, a differential operator taking values in the bundle of half-volume forms. This operator furnishes a short resolution of the structure sheaf, making possible a Quillen theory of determinant line bundles. Finally we show that the moduli space of super Riemann surfaces is embedded in the larger space of complex curves of dimension $1|1$.

1. Introduction

To describe a string moving through spacetime we introduce a two-dimensional parameter space X and consider functions from X to spacetime. Since the local dynamics of the string should depend only on the image of X in spacetime, we require that the action functional describing the string be independent of the parametrization chosen. Locally it should also not depend on any extra information describing the auxiliary space X, for instance a metric or connection on X.

When we move beyond the naive picture of strings above to the more sophisticated picture of an arbitrary conformal field theory, X takes on a more central role. Nevertheless the point made above are still valid: the classical action (when it exists) must be an intrinsically-defined functional of fields on X, defined without any use of the local shape of X. On the other hand, as is well known a smooth 2-manifold does not have enough information to permit us to define such