Commun. Math. Phys. 116, 529–538 (1988)

Communications in Mathematical Physics

Non-Integrability of the Truncated Toda Lattice Hamiltonian at Any Order

Haruo Yoshida*

Centre de Physique Théorique, Ecole Polytechnique, F-91128, Palaiseau, France

Abstract. Two sufficient conditions for the non-existence of an additional analytic integral are given for Hamiltonian systems with non-homogeneous polynomial potential of an arbitrary degree. An application is made to the truncated three-particle Toda lattice, which is proved to be non-integrable at any order.

1. Introduction and the Main Result

The so-called three-particle periodic Toda lattice [10] is defined by the Hamiltonian

$$H=(1/2)(p_x{}^2+p_y{}^2)+e^{\sqrt{3}x+y}+e^{-\sqrt{3}x+y}+e^{-2y}, \tag{1.1}$$

and integrable because of the existence of a second analytic integral [4, 6]

$$\Phi=(1/3)p_x(p_x{}^2-3p_y{}^2)+(p_x-\sqrt{3}p_y)e^{\sqrt{3}x+y}+(p_x+\sqrt{3}p_y)e^{-\sqrt{3}x+y}-2p_xe^{-2y}. \tag{1.2}$$

We now truncate the Taylor series of exponential functions in (1.1) at a finite order [3], i.e., consider the potential

$$V_N=\sum_{k=1}^{N}\{(\sqrt{3}x+y)^k+(-\sqrt{3}x+y)^k+(-2y)^k\}/k!. \tag{1.3}$$

V_2 is the harmonic oscillator and V_3 is identified with the so-called Hénon–Heiles potential [5],

$$V_3=(1/2)(x^2+y^2)+x^2y-(1/3)y^3, \tag{1.4}$$

after a proper change of scale. In the process of proving the integrability of the Toda lattice [4, 6], the exponential function plays a crucial role. Therefore, by truncating the exponential function to a finite order polynomial, we can no more expect the existence of an additional integral like (1.2). Indeed the main result of this paper is to prove

* Present address: Department of Mathematics, Imperial College of Science and Technology, 180 Queen's Gate, London SW7 2BZ, United Kingdom