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On a Theorem of Deift and Hempel*

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Abstract. We provide an alternative proof of the main result of Deift and Hempel [1] on the existence of eigenvalues of ν-dimensional Schrödinger operators $H_\lambda = H_0 + \lambda W$ in spectral gaps of H_0.

In a beautiful paper, Deift and Hempel [1] proved the existence of eigenvalues of Schrödinger operators $H_\lambda = H_0 + \lambda W$ in spectral gaps of H_0. For the relevance of this result to the theory of the color of crystals, see [1] and the references therein. In this note, we present an alternative proof of their main Theorem 1. We present our proof because of its striking simplicity.

Our hypotheses read:

(H.1) $V \in L^\infty(\mathbb{R}^\nu)$ real-valued, $\nu \in \mathbb{N}$.

(H.2) $W \in L^\infty(\mathbb{R}^\nu)$ real-valued, supp(W) compact, $W_-(x) \geqq 1$ for

$$x \in B_{\varepsilon_0}(x_0) := \{x \in \mathbb{R}^\nu \,|\, |x - x_0| < \varepsilon_0\} \quad \text{for some} \quad x_0 \in \operatorname{supp}(W_-)$$

and some $\varepsilon_0 > 0$ (here $W_\pm(x) := [|W(x)| \pm W(x)]/2$).

Given (H.1) and (H.2) we define in $L^2(\mathbb{R}^\nu)$ the Schrödinger operators

$$H_0 = -\Delta + V, H_\lambda = H_0 + \lambda W, \lambda \geqq 0 \tag{1}$$

with Δ the Laplacian defined on the standard Sobolev space $H^{2,2}(\mathbb{R}^\nu)$. Without loss of generality, we next modify $W_\pm$ to $\tilde{W}_\pm$ so that

(α) $0 \leqq \tilde{W}_\pm \in L^\infty(\mathbb{R}^\nu)$, supp$(\tilde{W}_\pm)$ compact,

(β) $W = W_+ - W_- = \tilde{W}_+ - \tilde{W}_-$,

(γ) supp$(\tilde{W}_+) = \{x \in \mathbb{R}^\nu \,|\, \varepsilon \leqq |x - x_0| \leqq R\} := \Sigma$, where R is chosen so large that

$$\operatorname{supp}(W) \subset B_R(x_0),$$

and where $\varepsilon \leqq \varepsilon_0$ as well as R will be chosen later. Moreover $\tilde{W}_+ \geqq 1$ on Σ.

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