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The $N^{7/5}$ Law for Charged Bosons

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Abstract. Non-relativistic bosons interacting with Coulomb forces are unstable, as Dyson showed 20 years ago, in the sense that the ground state energy satisfies $E_0 \leqq -AN^{7/5}$. We prove that 7/5 is the correct power by proving that $E_0 \geqq -BN^{7/5}$. For the non-relativistic bosonic, one-component jellium problem, Foldy and Girardeau showed that $E_0 \leqq -CN\rho^{1/4}$. This 1/4 law is also validated here by showing that $E_0 \geqq -DN\rho^{1/4}$. These bounds prove that the Bogoliubov type paired wave function correctly predicts the order of magnitude of the energy.

I. Introduction and Background

Twenty years ago Dyson and Lenard [5] proved the stability of ordinary non-relativistic matter with Coulomb forces, namely that the ground state energy, E_0, of an N-particle system satisfies $E_0 \geqq -A_1 N$ for some universal constant A_1. In ordinary matter, the negative particles (electrons) are fermions. At the same time, Dyson [4] proved that bosonic matter is definitely not stable; if all the particles (positive as well as negative) are bosons then $E_0 \leqq -A_2 N^{7/5}$ for some $A_2 > 0$. Dyson and Lenard [5] did prove, however, that $E_0 \geqq -A_3 N^{5/3}$ in the boson case, and thus the open problem was whether the correct exponent for bosons is 5/3 or 7/5 or something in between.

In this paper we prove that the $N^{7/5}$ law is correct for bosons by obtaining a lower bound $E_0 \geqq -A_4 N^{7/5}$. As is well known, the bosonic energy is the absolute lowest energy when no symmetry restriction is imposed.

It may appear that the difference between 5/3 and 7/5 is insignificant, especially since bosonic matter does not exist experimentally, but that impression would fail to take into account the essential difference between the ground states implied by

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