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Monopoles, Non-Linear σ Models, and Two-Fold Loop Spaces

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Abstract. In this paper we study the topology of $\hat{\mathcal{M}}_k$, the moduli spaces of $SU(2)$ monopoles associated with the Yang–Mills–Higgs and Bogomol'nyi equations, and $\mathcal{H}(m)_k$, non-linear σ models from quantum field theory. Beautiful work of Donaldson [18, 19], Hitchin [24, 25] and Taubes [37, 39, 40] shows that gauge equivalence classes of monopoles correspond to based rational self-maps of the Riemann sphere. Similarly, the non-linear σ models we consider here are based harmonic maps from the Riemann sphere to complex projective m space. In seminal work, Segal [35] studied $\mathcal{R}(m)_k$, the space of based rational maps from the Riemann sphere to complex projective m space of a fixed degree k. Any element of $\mathcal{R}(m)_k$ is clearly an element of $\Omega^2_k CP(m)$, the space of all based continuous maps from the Riemann sphere to complex projective m space of a fixed degree k, and this assignment gives rise to the natural inclusion of $\mathcal{R}(m)_k$ in $\Omega^2_k CP(m)$. Segal showed that these natural inclusions are homotopy equivalences through dimension $k(2m-1)$. As the topology of the two-fold loop space $\Omega^2 CP(m)$ is well understood, Segal's result gives a very efficient way to explicitly determine the low dimensional topology of $\mathcal{R}(m)_k$. Thus iterated loop spaces have much to say about the topology of monopoles and non-linear σ models.

In this paper we apply the theory of iterated loop spaces (more precisely, May's C_2 operad spaces [31]) to study $\hat{\mathcal{M}}_k$, $\mathcal{H}(m)_k$ and $\mathcal{R}(m)_k$. Our main technical device is to place a C_2 operad structure on these spaces which is compatible with the usual C_2 operad structure on $\Omega^2 CP(m)$. This will enable us to study the topology of $\mathcal{R}(m)_k$ and thus the topology of $\hat{\mathcal{M}}_k$ and $\mathcal{H}(m)_k$ above the range of the Segal equivalence.

The C_2 operad structure we define here on $\mathcal{R}(m)$ is very similar to the C_4 operad structure defined on the moduli spaces for instantons in [10]. It is worth recalling

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