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Holomorphic Coordinates for Supermoduli Space

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Abstract. Two-dimensional supergravity provides complex coordinates for supermoduli space compatible with its natural complex structure. Such coordinates are useful for investigating questions of holomorphic factorization of superdeterminants. They also demonstrate explicitly that the complex structure on supermoduli space is indeed integrable.

1. Introduction

In the first quantized theory of bosonic strings, string amplitudes are expressed as integrals over the space of all Riemann surfaces. The integrands of such integrals satisfy a remarkable holomorphy property, which serves as the basis for applications of algebraic geometry to strings.[1] Namely, the moduli space $\mathcal{M}$ of all Riemann surfaces has a natural complex structure, so that it makes sense to ask whether functions, forms, and so on are holomorphic on this space. The regulated determinants of $2d$ quantum field theory, while a priori only smooth functions of the moduli, occur in a combination with a special relationship to the complex structure [2, 3]. We can state this property precisely as follows. If we take the string measure μ and divide by

$$\mu_0 \equiv (\det \operatorname{Im} \tau)^{-13} |\psi^1 \wedge \ldots \wedge \psi^{3g-3}|^2 , \tag{1.1}$$

then the result is locally the absolute value squared of a holomorphic function on $\mathcal{M}$. Here τ is the period matrix of the Riemann surface in some marking and $\{\psi^i\}$ are a basis of quadratic differentials chosen to vary holomorphically. Given another such basis, or a different marking, (1.1) changes by the absolute square of some holomorphic function, so our statement of the holomorphy property is well-defined.

To establish that $\mathcal{M}$ has an integrable complex structure one first singles out certain tangent directions to $\mathcal{M}$ as holomorphic, and then shows that the tangent

[1] For a review see [1]