Commun. Math. Phys. 114, 645–685 (1988)

Communications in
Mathematical
Physics

String Theory and Algebraic Geometry of Moduli Spaces

Dirk-Jan Smit

Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80006, 3508 TA Utrecht, The Netherlands

Abstract. It is shown how the algebraic geometry of the moduli space of Riemann surfaces entirely determines the partition function of Polyakov's string theory. This is done by using elements of Arakelov's intersection theory applied to determinants of families of differential operators parametrized by moduli space. As a result we write the partition function in terms of exponentials of Arakelov's Green functions and Faltings' invariant on Riemann surfaces. Generalizing to arithmetic surfaces, i.e. surfaces which are associated to an algebraic number field K, we establish a connection between string theory and the infinite primes of K. As a result we conjecture that the usual partition function is a special case of a new partition function on the moduli space defined over K.

1. Introduction and Summary

Recently it has become clear that Polyakov's formulation for quantizing a string theory [38] has a profound geometrical interpretation. In this formulation one considers the string partition function Z, expressed as a perturbation series over random surfaces together with an integration over the space of metrics M_p of a Riemann surface M of genus p, and an integration over the space $\mathscr{E}$, containing all embeddings of the surface into d-dimensional space-time:

$$x : M \to \mathbb{R}^d \; x \in \mathscr{E} , \tag{1.1}$$

$$Z = \sum_{p=0}^{\infty} \int_{M_p \times \mathscr{E}} dg\, dx \exp(-S[x, g]),\; g \in M_p . \tag{1.2}$$

We assume that both the two-dimensional worldsheet swept out by the string, as well as the d-dimensional space-time in which the string moves can be Wick-rotated to Euclidean spaces. Furthermore, we shall restrict ourselves to closed strings so that we will be dealing exclusively with closed Riemann surfaces.