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On Reproducing Kernels and Quantization of States

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Abstract. Quantization of a mechanical system with the phase space a Kähler manifold is studied. It is shown that the calculation of the Feynman path integral for such a system is equivalent to finding the reproducing kernel function. The proposed approach is applied to a scalar massive conformal particle interacting with an external field which is described by deformation of a Hermitian line bundle structure.

1. Introduction

In the case of ordinary quantum mechanics the space of pure states is a projective complex Hilbert space. As a consequence the role of complex numbers is crucial in the description of quantum phenomena. On the other hand, the classical mechanical systems are described in terms of real differential geometry. However, many leading quantized classical systems have complex differential manifolds as phase spaces. Let us give a few examples: 1) the space of orbits of the n-dimensional harmonic isotropic oscillator is $\mathbb{CP}(n-1)$ (see [6]); 2) the phase space of a spin system is given by $\mathbb{CP}(1)$; 3) $\mathbb{CP}(1)\times\mathbb{CP}(1)$ is the phase space of orbits corresponding to the negative energy level in the Kepler problem (see [16]). The twistor theory provides us also with a wide class of complex phase spaces. In general they are realized as the orbits of the conformal group on twistor flag spaces (see [10]). In particular, the space of positive projective twistors is the phase of the photon with positive helicity (see [13, 14]). Finally one should emphasize the important role of the Bargmann-Fock-Segal representation in quantum mechanics (because of its holomorphicity).

In Sect. 2 of this paper we study the quantization of a classical mechanical system where the phase space M is a Kähler manifold. The basic feature which distinguishes such a system among the others is the possibility of quantization of classical states. This means that in some special case, when the Hilbert space of quantum states satisfies some condition of ampleness (see Propositions 2 and 3), one can embed M into $\mathbb{CP}(\mathscr{M})$, where $\mathscr{M}$ consists of square integrable holomorphic sections of a Hermitian line bundle $\mathbb{E}$ over M. Using this embedding one