Commun. Math. Phys. 114, 345–349 (1988)

Communications in
Mathematical
Physics

A Variational Expression for the Relative Entropy

Dénes Petz

Mathematical Institute of HAS, Reàltanoda u. 13–15, H-1364 Budapest, PF.127, Hungary

Abstract. We prove that for the relative entropy of faithful normal states φ and ω on the von Neumann algebra M the formula

$$S(\varphi, \omega) = \sup\{\omega(h) - \log \varphi^h(I) \colon h = h^* \in M\}$$

holds.

In general von Neumann algebras the relative entropy was defined and investigated by Araki [1, 3]. After Lieb had proved the joint convexity of the relative entropy in the type I case [10] several proofs appeared in the literature and they all benefited from the operator convexity of the function $t \to -\log t$ [8, 11]. Improving a result of Pusz and Woronowicz [14] Kosaki [9] obtained a variational formula for the relative entropy, which allows to extend the notion also to C^*-algebras. The expression we are going to deal with is of a different kind. It shows that the relative entropy $S(\varphi, \omega)$ as a function of φ is the conjugate convex function (i.e., Legendre transform) of the convex function $h \to \log \varphi^h(I)$, where φ^h denotes the inner perturbation of the state φ by the selfadjoint operator h. The perturbed state φ^h was used by Araki to extend the Golden-Thompson inequality ([7, 18], see also [15]) to traceless von Neumann algebras. Approaching our variational expression for the relative entropy we generalize the Golden-Thompson-Araki inequality [2] essentially and we state also the exact condition for the equality.

If φ and ω are faithful normal states of the von Neumann algebra M then the relative entropy is defined by means of the relative modular operator $\Delta(\varphi, \omega)$. If Ω is the vector representative of ω in the natural positive cone P then

$$S(\varphi, \omega) = -\langle \log \Delta(\varphi, \omega)\Omega, \Omega \rangle .$$

The variational expression of Kosaki says that

$$S(\varphi, \omega) = \sup \sup \left\{ \log n - \int_{1/n}^{\infty} t^{-1}\omega(y(t)^* y(t)) + t^{-2}\varphi(x(t)x(t)^*)dt \right\},$$