Commun. Math. Phys. 114, 69–91 (1988)

Communications in Mathematical Physics

Systems with Outer Constraints. Gupta–Bleuler Electromagnetism as an Algebraic Field Theory

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Abstract. Since there are some important systems which have constraints not contained in their field algebras, we develop here in a C^*-context the algebraic structures of these. The constraints are defined as a group G acting as outer automorphisms on the field algebra $\mathscr{F}$, $\alpha: G \mapsto \mathrm{Aut}\,\mathscr{F}$, $\alpha_G \not\subset \mathrm{Inn}\,\mathscr{F}$, and we find that the selection of G-invariant states on $\mathscr{F}$ is the same as the selection of states ω on $M(G \underset{\alpha}{\times} \mathscr{F})$ by $\omega(U_g) = 1\ \forall g \in G$, where $U_g \in M(G \underset{\alpha}{\times} \mathscr{F}) \backslash \mathscr{F}$ are the canonical elements implementing α_g. These states are taken as the physical states, and this specifies the resulting algebraic structure of the physics in $M(G \underset{\alpha}{\times} \mathscr{F})$, and in particular the maximal constraint free physical algebra $\mathscr{R}$. A nontriviality condition is given for $\mathscr{R}$ to exist, and we extend the notion of a crossed product to deal with a situation where G is not locally compact. This is necessary to deal with the field theoretical aspect of the constraints. Next the C^*-algebra of the CCR is employed to define the abstract algebraic structure of Gupta–Bleuler electromagnetism in the present framework. The indefinite inner product representation structure is obtained, and this puts Gupta–Bleuler electromagnetism on a rigorous footing. Finally, as a bonus, we find that the algebraic structures just set up, provide a blueprint for constructive quadratic algebraic field theory.

1. Introduction

A degenerate system is defined as having a nonphysical degree of freedom, and is usually characterised by supplementary conditions, or by the action of a gauge group on it. The physicist has the task of extracting the physical subsystem from such a degenerate one. Indeed, physical information such as boundary conditions or constraints, is often injected into a theory through the use of supplementary conditions, and one could argue (as is done in [3]), that in a Lagrangian framework the field equations should also be imposed as supplementary conditions on the field algebra.

In algebraic field theory, a system is described by a unital C^*-algebra $\mathscr{F}$ as the field algebra, together with its set of states $\wp$, and hence a supplementary condition can be imposed either on $\mathscr{F}$, or on $\wp$, called respectively algebraic and state