Commun. Math. Phys. 114, 1–53 (1988)

Communications in
Mathematical
Physics

Coadjoint Orbits of the Virasoro Group

Edward Witten*

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Abstract. The coadjoint orbits of the Virasoro group, which have been investigated by Lazutkin and Pankratova and by Segal, should according to the Kirillov-Kostant theory be related to the unitary representations of the Virasoro group. In this paper, the classification of orbits is reconsidered, with an explicit description of the possible centralizers of coadjoint orbits. The possible orbits are diff(S^1) itself, diff(S^1)/S^1, and diff(S^1)/$SL^{(n)}(2,R)$, with $SL^{(n)}(2,R)$ a certain discrete series of embeddings of $SL(2,R)$ in diff(S^1), and diffS^1/T, where T may be any of certain rather special one parameter subgroups of diffS^1. An attempt is made to clarify the relation between orbits and representations. It appears that quantization of diffS^1/S^1 is related to unitary representations with nondegenerate Kac determinant (unitary Verma modules), while quantization of diff$S^1/SL^{(n)}(2,R)$ is seemingly related to unitary representations with null vectors in level n. A better understanding of how to quantize the relevant orbits might lead to a better geometrical understanding of Virasoro representation theory. In the process of investigating Virasoro coadjoint orbits, we observe the existence of left invariant symplectic structures on the Virasoro group manifold. As is described in an appendix, these give rise to Lie bialgebra structures with the Virasoro algebra as the underlying Lie algebra.

1. Introduction

The representation theory of finite dimensional compact semi-simple Lie groups is greatly clarified by the Borel-Weil-Bott theorem, whose content is roughly as follows. Let G be a compact semi-simple Lie group, and T a maximal torus. The quotient G/T (defined by the equivalence relation $g \sim gt$, with $g \in G$, $t \in T$) is a compact, complex manifold. The theorem in question interprets the irreducible unitary representations of G as spaces of holomorphic sections of certain holomorphic line bundles over G/T.

* Research supported in part by NSF grants PHY 80-19754 and 86-16129