Commun. Math. Phys. 113, 659–700 (1988)

Communications in
Mathematical
Physics

The Kowalewski and Hénon-Heiles Motions as Manakov Geodesic Flows on $SO(4)$ – a Two-Dimensional Family of Lax Pairs*

Mark Adler[1] and Pierre van Moerbeke[1,2]

[1] Department of Mathematics, Brandeis University, Waltham, MA 02254, USA
[2] Department of Mathematics, University of Louvain, Louvain-la-Neuve, Belgium

Abstract. The invariant surfaces for the Kowalewski top, the Hénon-Heiles system and the Manakov geodesic flow on $SO(4)$ complete into Abelian surfaces A, by adjoining, in each case, a divisor D of arithmetic genus 9; these divisors belong to the same linear system on A and they each define a polarization (2, 4). Therefore there are rational maps transforming the Kowalewski top and the Hénon-Heiles system into Manakov's geodesic flow on $SO(4)$. This paper deals with the precise geometric relationship between these three problems; it is based on the splitting of the 8-dimensional space of sections of D (theta-functions) into an even and an odd part and also on a normal form for the six quadrics describing A, as embedded in $\mathbb{P}^7$. As a byproduct, we get a 2-dimensional family of Lax pairs for both the Kowalewski top and the Hénon-Heiles system.

1. Introduction

Integrable systems have been integrated classically in terms of quadratures, usually through a sequence of very ingenious algebraic manipulations especially tailored to the problem. More recently, it was realized that whenever a system could be represented as a family of Lax pairs – often arising in the context of coadjoint orbits of Kac-Moody Lie algebras – the system could be linearized on the Jacobian of a spectral curve, defined by the characteristic polynomial of one of the matrices in the Lax pair. However this approach has remained unsatisfactory; indeed (i) finding such families of Lax pairs often requires just as much ingenuity and luck as to actually solve the problem; (ii) it often conceals the actual geometry of the problem. Therefore we have engaged in a systematic approach towards solving integrable systems, based on the Laurent solutions of the differential equations [5]; This is done in the context of *algebraically completely integrable systems*. The latter means: the system has polynomial invariants, in sufficient

* The support of a National Science Foundation grant #84-03136 is gratefully acknowledged