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Quantum Field Theory, Grassmannians, and Algebraic Curves

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Abstract. This paper is devoted in part to clarifying some aspects of the relation between quantum field theory and infinite Grassmannians, and in part to pointing out the existence of a close analogy between conformal field theory on Riemann surfaces and the modern theory of automorphic representations. Along the way we develop a multiplicative analog of the usual additive Ward identities of current algebra. We also reformulate the additive Ward identities in a way which may be useful, in terms of the residues of operator-valued differential forms. A concluding section is devoted to some remarks on string field theory. In an appendix, we attempt to clarify the recent construction by Beilinson, Manin, and Schechtman of what might be called global Virasoro algebras.

The present paper consists of several sections. In Sects. (1) and (2), I will attempt to describe in physical terminology some aspects of the relation, surveyed in [1], between Riemann surfaces and infinite Grassmannians. This relation has been essential in recent studies of the Schottky problem [2, 3], and its relation with quantum field theory and string theory have been the subject of recent discussions [4–6] from a physical viewpoint. In the first section we will consider the Grassmannian of [1] as the space of boundary conditions on the $\bar{D}$ operator. This way of looking at things really provides the essential link between Grassmannians and the theory of free fermions. In the second we introduce "multiplicative Ward identities." These are needed to describe the relation between the Baker function and the tau function. They also, I believe, shed considerable light on the whole phenomenon of bosonization of fermions. And they are a needed preliminary for the latter part of the paper.

In the third section, we reformulate the Ward identities of conformal field theory, first described in [14], in terms of "operator valued differential forms," which will have already made their appearance in Sect. one. In particular, we will

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