Commun. Math. Phys. 113, 299–339 (1987)

Communications in
Mathematical
Physics

The Decorated Teichmüller Space of Punctured Surfaces

R.C. Penner*

Math Department, University of Southern California, Los Angeles, CA 90089, USA

Abstract. A principal $\mathbb{R}_+^s$-bundle over the usual Teichmüller space of an s times punctured surface is introduced. The bundle is mapping class group equivariant and admits an invariant foliation. Several coordinatizations of the total space of the bundle are developed. There is furthermore a natural cell-decomposition of the bundle. Finally, we compute the coordinate action of the mapping class group on the total space; the total space is found to have a rich (equivariant) geometric structure. We sketch some connections with arithmetic groups, diophantine approximations, and certain problems in plane euclidean geometry. Furthermore, these investigations lead to an explicit scheme of integration over the moduli spaces, and to the construction of a "universal Teichmüller space," which we hope will provide a formalism for understanding some connections between the Teichmüller theory, the KP hierarchy and the Virasoro algebra. These latter applications are pursued elsewhere.

Let F_g^s denote the genus g surface with s points removed, where $2g-2+s>0$, $g \geqq 0$, and $s \geqq 1$. This paper presents a number of results on the Teichmüller space $\mathscr{T}_g^s$ of marked conformal classes of complete finite-area metrics on F_g^s. Actually, we define a principal $\mathbb{R}_+^s$ foliated fibration $\phi: \tilde{\mathscr{T}}_g^s \to \mathscr{T}_g^s$, where the fiber over a point of $\mathscr{T}_g^s$ is the space of all horocycles about the punctures of F_g^s; the total space of the fibration is called the "decorated Teichmüller space." The mapping class group MC_g^s of homotopy classes of orientation-preserving homeomorphisms of F_g^s (which may permute the punctures) acts on $\mathscr{T}_g^s$ and $\tilde{\mathscr{T}}_g^s$, and the map ϕ is equivariant. The theory described below is developed for the decorated Teichmüller space $\tilde{\mathscr{T}}_g^s$, and the analogous results for $\mathscr{T}_g^s$ itself are discussed in an addendum.

Our first result gives a homeomorphism between $\tilde{\mathscr{T}}_g^s$ and $\mathbb{R}_+^q$, $q=6g-6+3s$. Specifically, we assign a positive real number $\lambda(c;\tilde{\Gamma}_m)$ to $\tilde{\Gamma}_m \in \tilde{\mathscr{T}}_g^s$ and an isotopy class c of arc in F_g^s connecting punctures; fixing a family Δ of such arcs so that each

* Partially supported N.S.F. grant DMS 8601162