Commun. Math. Phys. 113, 155–172 (1987)

Communications in
Mathematical
Physics

Witten's Gauge Field Equations and an Infinite-Dimensional Grassmann Manifold

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Abstract. Witten's gauge fields are interpreted as motions on an infinite-dimensional Grassmann manifold. Unlike the case of self-dual Yang-Mills equations in Takasaki's work, the initial data must satisfy a system of differential equations since Witten's equations comprise a pair of spectral parameters. Solutions corresponding to (anti-) self-dual Yang-Mills fields are characterized in the space of initial data and in application, some Yang-Mills fields which are not self-dual, anti-self-dual nor abelian can be constructed.

0. Introduction

Consider a gauge field ∇ in the eight-dimensional complex space $\mathbb{C}^8$ satisfying

$$[\nabla_{y_\mu}, \nabla_{y_\nu}] = (1/2) \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \varepsilon_{\mu\nu\alpha\beta} [\nabla_{y_\alpha}, \nabla_{y_\beta}] \ ,$$

$$[\nabla_{z_\mu}, \nabla_{z_\nu}] = (-1/2) \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \varepsilon_{\mu\nu\alpha\beta} [\nabla_{z_\alpha}, \nabla_{z_\beta}] \ ,$$

$$[\nabla_{y_\mu}, \nabla_{z_\nu}] = 0 \ , \quad (\mu, \nu = 0, 1, 2, 3) \ , \tag{0.1}$$

where $(y, z) = (y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3)$ are coordinates of $\mathbb{C}^8$, ∇_{y_μ} and ∇_{z_μ} are covariant derivatives, and $\varepsilon_{\mu\nu\alpha\beta}$ denotes the totally antisymmetric tensor such that $\varepsilon_{0123} = 1$.

Set $x = (y+z)/2$, $w = (y-z)/2$. Witten [9] pointed out that Eq. (0.1) imply the full Yang-Mills equations

$$\sum_{\mu=0}^{3} [\nabla_{x_\mu}, [\nabla_{x_\mu}, \nabla_{x_\nu}]] = 0 \qquad (\nu = 0, 1, 2, 3) \tag{0.2}$$

on the diagonal subspace $\Delta = \{(y, z) \in \mathbb{C}^8 | w = 0\}$, and further, that a gauge field on Δ satisfies (0.2) if and only if it can be extended to a neighborhood of Δ consistently to (0.1) mod $(w_0, w_1, w_2, w_3)^2$. Here $(w_0, w_1, w_2, w_3)^2$ denotes the square of the ideal generated by w_0, w_1, w_2, and w_3.