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Invariant Circles and Rotation Bands In Monotone Twist Maps

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Abstract. In this paper we show that the existence of certain orbits or minimal sets in an area-preserving monotone twist map is necessary and sufficient for the non-existence of invariant circles with specified rotation numbers. The necessity of these conditions follows from classic results of Birkhoff and recent results of Mather. The sufficiency of these conditions depends on the notion of a rotation band which associates a set of rotation numbers with a given orbit or invariant set. We also make some remarks on Mather's paper [M4]. In particular, we use his main theorem to give a lower bound on the width of the interval of rotation numbers associated with the zone of instability that "contains" the irrational ω when f has no invariant circle with rotation number ω.

Section 0

In this paper we generalize some results of [B-H] by showing that the existence of certain orbits or minimal sets in an area-preserving monotone twist map of the annulus is necessary and sufficient for the non-existence of invariant circles with specified rotation numbers. The necessity of these conditions follows from classic results of Birkhoff ([B1, B2 and B3]) and recent results of Mather ([M4]). The sufficiency of these conditions depends on the notion of the rotation band which is defined in Sect. 2. The rotation band of an invariant set is an open interval of real numbers whose endpoints, roughly speaking, quantify the fastest and slowest rate of rotation associated with the set. It can be viewed as a generalization of Birkhoff's definition of an inner and outer rotation number associated with an invariant set that separates the annulus ([B3]).

The main lemma (Lemma 2) states that an area preserving monotone twist maps has no invariant circle whose rotation number lies in the rotation band of an orbit or minimal set. If f preserves the angular order on an invariant set, its rotation band is empty. Thus the non-existence of invariant circles is seen to be

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