Commun. Math. Phys. 113, 1–26 (1987)

Communications in Mathematical Physics



# The A-D-E Classification of Minimal and $A_1^{(1)}$ Conformal Invariant Theories

A. Cappelli*, C. Itzykson, and J. B. Zuber

Service de Physique Théorique, CEN-Saclay, F-91191 Gif-sur-Yvette Cedex, France

**Abstract.** We present a detailed and complete proof of our earlier conjecture on the classification of minimal conformal invariant theories. This is based on an exhaustive construction of all modular invariant sesquilinear forms, with positive integral coefficients, in the characters of the Virasoro or of the $A_1^{(1)}$ Kac-Moody algebras, which describe the corresponding partition functions on a torus. A remarkable correspondence emerges with simply laced Lie algebras.

## I. Introduction

1. The minimal conformal invariant models describe a class of massless two dimensional field theories, with known critical properties [1]. Their anomalous dimensions and operator content are encoded in the expression of the partition function on a torus. The sum over states decomposes into pairs of irreducible representations of the Virasoro algebra, with central charge $c$ rational and smaller than 1, yielding a sesquilinear form in the characters $\chi_h$,

$$Z(\tau)=\sum \mathcal{N}_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}^*(\tau) \ .$$

In this formula $\tau$ is the ratio of the two periods on the torus, and the summation extends over a finite table of known $(h,\bar{h})$ values. The non-negative integral coefficients $\mathcal{N}_{h,\bar{h}}$ yield the multiplicities of primary scaling operators $\varphi_{h,\bar{h}}$, which are in one to one correspondence with the products $\chi_h\chi_{\bar{h}}^*$ of characters. Cardy [2] noticed that modular invariance is a consistency condition on these partition functions.

Our aim here is to present a detailed proof of the classification of these positive modular invariants, announced in [3]. As these theories describe statistical models at criticality, this classifies the universality classes of two dimensional critical phenomena, pertaining to $c<1$, with finitely many primary observables. They include for instance the Ising and three-state Potts models.

* Permanent address: Istituto Nazionale di Fisica Nucleare, Sezione di Firenze, Largo E. Fermi 2, I-50125 Firenze, Italy