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Dynamical Entropy of C^* Algebras and von Neumann Algebras

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Abstract. The definition of the dynamical entropy is extended for automorphism groups of C^* algebras. As an example, the dynamical entropy of the shift of a lattice algebra is studied, and it is shown that in some cases it coincides with the entropy density.

Introduction

While in the 19th century the concept of entropy appeared in thermodynamics and its connection with statistics was realized, a satisfactory mathematical theory became available only through the Kolmogorov-Sinai entropy of automorphisms, which was a byproduct of progresses in information theory [1].

Not only did this new mathematical concept of a dynamical entropy for a measure invariant under a transformation clarify the mathematical set-up of thermodynamics, especially because it allowed a formulation of variational principles [2] without appealing again and again to a thermodynamic limit, it became also the key notion in ergodic theory. Through the work of Bowen and Ruelle the thermodynamic formulation has invaded the theory of smooth dynamical systems and appeared to be a crucial tool for problems such as the iteration of fractional transformations in C [3].

From the early beginning of the work of Kolmogorov and Sinai it was clear that a quantum or non-commutative analogue was required for both to be applicable in microphysics and to provide an important mathematical concept which in fact von Neumann and Sinai were asking for. For instance the work of Cuntz and Krieger on subshifts of finite type leads to a natural non-commutative C^*-algebra together with an automorphism for which entropy and variational inequalities would be very relevant. As far as quantum thermodynamics is concerned, there exists a definition of entropy density [4] but it refers in a crucial manner to a net of finite subsystems and thus has no a priori invariance properties as the KS entropy does. More precisely, in the classical context a corollary of the KS theorem shows that the entropy density computed on a limit of finite systems is