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Asymptotic Inverse Spectral Problem for Anharmonic Oscillators

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Dedicated to V. P. Gurarii on his 50^{th} birthday

Abstract. We study perturbations $L=A+B$ of the harmonic oscillator $A=\frac{1}{2}(-\partial^2+x^2-1)$ on $\mathbb{R}$, when potential $B(x)$ has a prescribed asymptotics at ∞, $B(x)\sim|x|^{-\alpha}V(x)$ with a trigonometric even function $V(x)=\sum a_m\cos\omega_m x$. The eigenvalues of L are shown to be $\lambda_k=k+\mu_k$ with small $\mu_k=O(k^{-\gamma})$, $\gamma=1/2+1/4$.

The main result of the paper is an asymptotic formula for spectral fluctuations $\{\mu_k\}$,

$$\mu_k\sim k^{-\gamma}\tilde{V}(\sqrt{2k})+c/\sqrt{2k} \quad \text{as} \quad k\to\infty,$$

whose leading term $\tilde{V}$ represents the so-called "Radon transform" of V,

$$\tilde{V}(x)=\text{const}\sum\frac{a_m}{\sqrt{\omega_m}}\cos(\omega_m x-\pi/4).$$

as a consequence we are able to solve explicitly the inverse spectral problem, i.e., recover asymptotic part $|x|^{-\alpha}V(x)$ of B from asymptotics of $\{\mu_k\}_1^\infty$.

The standard spectral problem for a perturbation $L=A+B$ of a differential operator A with the given spectrum $\{\lambda_k(A)\}_1^\infty$ asks to (approximately) calculate the eigenvalues of L in terms of $\{\lambda_k(A)\}$ and the perturbation. For a "relatively small" perturbation B, the k^{th} eigenvalue of L is

$$\lambda_k(L)=\lambda_k(A)+\mu_k,$$

so one is asked to calculate spectral fluctuations $\{\mu_k\}_1^\infty$. The corresponding inverse problem is then to recover $B(x)$ from the given (admissible) sequence of eigenvalues $\{\lambda_k(\mathrm{L})\}_1^\infty$ or fluctuations $\{\mu_k\}_1^\infty$.

Spectral problems were extensively studied in various contexts for both ordinary and partial differential operators. The best known example is the regular Sturm-Liouville problem: $L=\frac{d^2}{dx^2}+V(x)$ on $[0,1]$. The old result of Borg [Bo]