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Fermions and Octonions

P. Goddard[1], W. Nahm[2], D.I. Olive[3], H. Ruegg[4], and A. Schwimmer[5]

[1] DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom
[2] Institute for Theoretical Physics, SUNY at Stony Brook, NY 11794, USA
[3] The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom
[4] Département de Physique Théorique, University of Genèva, Switzerland
[5] Department of Physics, Weizmann Institute, Rehovot, Israel

Abstract. We analyse further the algebraic structure of dependent fermions, namely ones interrelated by the vertex operator construction. They are associated with special sorts of lattice systems which are introduced and discussed. The explicit evaluation of the relevant cocycles leads to the result that the operator product expansion of the fermions is related in a precise way to one or another of the division algebras given by complex numbers, quaternions or octonions. The latter case is seen to be realised in the light cone formalism of superstring theory.

1. Introduction

It has emerged that free massless Majorana fermions in a two-dimensional spacetime play an important role in various problems in theoretical physics controlled by conformal symmetry. In the fermionic string theory [1], multiplets of such fields occur which transform vectorially either under the Lorentz group $SO(9,1)$ or its light cone restriction $SO(8)$. In the heterotic string theory [2] there are fermion fields forming linear or non-linear representations of the gauge group [3]. In statistical physics, such fermions give a convenient description of certain two-dimensional models [4]. The Ising model can be discussed in terms of one such field [5]. Fermion fields also arise in the theory of solitons [6, 7].

The simplest situation occurs when the various real component fermi fields ψ_i are "independent" in that they anticommute when their suffices differ. This statement can be expressed via the operator product expansion as

$$\psi_i(z)\psi_j(\zeta)=\frac{z\delta_{ij}}{z-\zeta}+\text{regular}\ ,\qquad |z|>|\zeta|\ ,\tag{1.1}$$

using notation common in string theory (see the review [8]). This framework is insufficiently general to encompass all situations of physical interest. For example, when Green and Schwarz [9] reformulated superstring theory [10] so as to render manifest the space-time supersymmetry, at least in the light cone gauge, they