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Kähler-Einstein Metrics on Complex Surfaces with $C_1 > 0$

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Dedicated to Walter Thirring on his 60th birthday

Abstract. Various estimates of the lower bound of the holomorphic invariant $\alpha(M)$, defined in [T], are given here by using branched coverings, potential estimates and Lelong numbers of positive, d-closed $(1,1)$ currents of certain type, etc. These estimates are then applied to produce Kähler-Einstein metrics on complex surfaces with $C_1 > 0$, in particular, we prove that there are Kähler-Einstein structures with $C_1 > 0$ on any manifold of differential type $CP^2 \# \overline{nCP^2}$ $(3 \leqq n \leqq 8)$.

The question of finding gravitational instantons has been important in mathematical physics. In this paper, we restrict ourselves to Kähler-Einstein metrics. In 1976, the second author solved Calabi's conjecture on the Kähler-Einstein metric. However, an important related question has not been solved yet. When a compact complex manifold has positive first Chern class, does it admit any Kähler-Einstein metric?

The theorem of Matsushima says that if such a metric exists, the automorphism group must be reductive. More recently, Futaki introduced more invariants related to the automorphism group and he demonstrated that these invariants are zero if the Kähler-Einstein metric exists. Some authors expressed the hope that if the automorphism group is discrete, then the Kähler-Einstein metric exists. However, there is another integrability condition, the tangent bundle of a Kähler-Einstein manifold has to be stable unless reducible. (The work of Bogomolov, Kobayashi, Lübke leads to such a conclusion.) Since the stability of the tangent bundle is more related to the linearized version of the equation, it is likely that a

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