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A New Proof of M. Herman's Theorem

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Dedicated to Walter Thirring on his 60th birthday

Abstract. A new proof of the M. Herman theorem on the smooth conjugacy of a circle map is presented here. It is based on the thermodynamic representation of dynamical systems and the study of the ergodic properties for the corresponding radom variables.

1. Introduction

In this paper we present a new proof of M. Herman's famous theorem about smooth conjugacy of diffeomorphisms of the circle to rotations. Our proof is based on a version of thermodynamic formalism which was used earlier for the study of Feigenbaum's mappings of the interval (see [1]) and later for some critical mappings of the circle (see [2, 3]).

Consider a strictly monotone continuous function φ such that $\varphi(x+1) = \varphi(x)+1$. It defines a homeomorphism of S^1 through the equation: $T_\varphi x = \{\varphi(x)\}$, $x \in [0;1)$. Denote the rotation number of T_φ by ϱ.

Assumptions. A.1. $\varphi \in C^{2+\gamma}$, $\gamma > 0$, $\varphi' \geqq \text{const} > 0$;

A.2. ϱ is irrational and if

$$\varrho = [k_1, k_2, \ldots, k_n, \ldots]$$

is the expansion of ϱ into continued fraction, then $k_n \leqq \text{const}\, n^\nu$, $\nu > 0$.

If ϱ is irrational and $\varphi \in C^2$, then Denjoy's classical theorem states that T_φ is topologically isomorphic to the rotation with angle ϱ. In other words there exists a strictly monotone continuous function ψ, $\psi(x+1) = \psi(x)+1$, such that $\psi(\varphi(x)) = \psi(x) + \varrho$. If we denote $R_\varrho = T_{\varphi_0}$ $\varphi_0 = x + \varrho$, then the last equality means $T_\psi T_\varphi = R_\varrho T_\psi$.

Herman's Theorem. *Under the Assumptions A.1, A.2 the function* $\psi \in C^1$.

For us it will be convenient to prove a statement equivalent to Herman's theorem. This is pointed out in Arnold's paper [11]. Let us write the equation for the density $\pi(x)$ of an invariant absolutely continuous measure, provided that such