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Uniqueness of the Infinite Cluster and Continuity of Connectivity Functions for Short and Long Range Percolation

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Abstract. For independent translation-invariant irreducible percolation models, it is proved that the infinite cluster, when it exists, must be unique. The proof is based on the convexity (or almost convexity) and differentiability of the mean number of clusters per site, which is the percolation analogue of the free energy. The analysis applies to both site and bond models in arbitrary dimension, including long range bond percolation. In particular, uniqueness is valid at the critical point of one-dimensional $1/|x-y|^2$ models in spite of the discontinuity of the percolation density there. Corollaries of uniqueness and its proof are continuity of the connectivity functions and (except possibly at the critical point) of the percolation density. Related to differentiability of the free energy are inequalities which bound the "specific heat" critical exponent α in terms of the mean cluster size exponent γ and the critical cluster size distribution exponent δ; e.g., $1+\alpha \leqq \gamma(\delta/2-1)/(\delta-1)$.

1. Introduction

In this paper, we present general results dealing with three related issues in percolation theory: a) uniqueness of the infinite cluster, b) continuity (in the natural parameters) of the connectivity functions and c) continuous differentiability of the "free energy" function. New relations are derived here between these three properties and used for the resolution of some old problems in the mathematical analysis of percolation. Specifically, the question of uniqueness of the infinite cluster is resolved (for independent, translation invariant models on finite dimensional lattices) by first proving uniqueness to be equivalent to either of the continuity statements b) or c)

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