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Variation of Discrete Spectra

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Abstract. A formula [see (1) below] estimating collectively the variation of eigenvalues of a symmetric matrix under a perturbation is extended to the case of discrete eigenvalues of a selfadjoint operator in Hilbert space, under the assumption that the perturbation is compact. For this purpose, the notion of an extended enumeration of discrete eigenvalues is introduced.

1. The following result is known (see [1, Theorem II-6.11]).

Theorem I. *If A, B are $n\times n$ hermitian matrices, their eigenvalues α_j and β_j can be enumerated, with multiple eigenvalues repeated according to the multiplicity, in such a way that for any real-valued convex function Φ on $\mathbb{R}$ we have*

$$\sum_j \Phi(\beta_j-\alpha_j)\leqq \sum_k \Phi(\gamma_k), \tag{1}$$

where the γ_k are the eigenvalues of $C=B-A$, similarly repeated.

In what follows we shall generalize (1), with slight modifications, to the infinite-dimensional case. To this end we introduce several definitions.

Let A be a selfadjoint operator in a separable Hilbert space H. An isolated point of the spectrum of A is automatically an eigenvalue; if it has finite multiplicity, we shall call it a *discrete eigenvalue* of A. The complement of the set of all discrete eigenvalues relative to the spectrum is the *essential spectrum.* The essential spectrum is closed in $\mathbb{R}$; its complement in $\mathbb{R}$ consists of at most countably many intervals I_n. Each discrete eigenvalue belongs to one of these intervals.

By an *extended enumeration of discrete eigenvalues* for A we mean a sequence $\{\alpha_j\}$ with the following properties. (a) Every discrete eigenvalue of A with multiplicity m appears exactly m-times in the values α_j. We refer to these values as *proper* values of the sequence. (b) All other values of the α_j, referred to as *improper values*, belong to the countable set consisting of all the boundary points of the intervals I_n stated above. Improper values may or may not be eigenvalues, and their number may be finite or infinite. If there are no improper values, we simply speak of an *enumeration.*