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Geometry of Skyrmions

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Abstract. A Skyrmion may be regarded as a topologically non-trivial map from one Riemannian manifold to another, minimizing a particular energy functional. We discuss the geometrical interpretation of this energy functional and give examples of Skyrmions on various manifolds. We show how the existence of conformal transformations can cause a Skyrmion on a 3-sphere to become unstable, and how this may be related to chiral symmetry breaking.

1. Introduction

A Skyrmion is a classical static field configuration of minimal energy in a non-linear scalar field theory. The scalar field is the pion field, and the Skyrmion represents a baryon. The Skyrmion has a topological charge which prevents it being continuously deformed to the vacuum field configuration. This charge is identified with the conserved baryon number which prevents a proton from decaying into pions [1]. The Skyrmion picture is in fair quantitative agreement with experimentally determined properties of protons and neutrons and their excited states [2].

Mathematically the Skyrmion is a topologically non-trivial map from physical 3-dimensional space S to a target manifold Σ [3]. The metrics on both S and Σ are essential and the energy of the Skyrmion is a measure of the geometrical distortion induced by the map. This is real "rubber-sheet" geometry. Indeed, the Skyrmion's energy is very like the strain energy of a deformed material in one version of non-linear elasticity theory [4], generalized to curved space. The Skyrmion may also be regarded as a generalized harmonic map [5].

In Sect. 2 we review the geometry of the strain tensor, and show that the Skyrme model's natural setting is Riemannian geometry. There is no need for the target manifold to be a Lie group, as is often assumed. In Sect. 3 we study maps between a domain and target which differ only by a constant scale factor. The identity map is always a stationary point of the energy functional in this situation, but it is not always stable. We illustrate this in Sect. 4 by the example of Skyrmions on a 3-sphere and on a 3-torus. The standard Skyrmion in flat space emerges when