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Kac-Moody Symmetry of Generalized Non-Linear Schrödinger Equations

A. D. W. B. Crumey

Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom

Abstract. The classical non-linear Schrödinger equation associated with a symmetric Lie algebra $\mathscr{g} = \mathscr{k} \oplus \mathscr{m}$ is known to possess a class of conserved quantities which form a realization of the algebra $\mathscr{k} \otimes \mathbb{C}[\lambda]$. The construction is now extended to provide a realization of the Kac-Moody algebra $\mathscr{k} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ (with central extension). One can then define auxiliary quantities to obtain the full algebra $\mathscr{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. This leads to the formal linearization of the system.

1. Introduction

This is a continuation of the work presented in [1], in which it was shown how to construct conserved quantities for the generalized non-linear Schrödinger (GNLS) equation of Fordy and Kulish [2]:

$$iq_t^\alpha = q_{xx}^\alpha \pm q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta} \tag{1.1}$$

(summation is implied over repeated indices) which is associated with a Lie algebra $\mathscr{g} = \mathscr{k} \oplus \mathscr{m}$. $q(x,t)$ is a matrix field in $1+1$ dimensions whose components lie in $\mathscr{m}$, and $\mathscr{k}$ is the centralizer of a special Cartan subalgebra element E satisfying the property

$$[E, e_\alpha] = -ie_\alpha \tag{1.2}$$

for all $e_\alpha \in \mathscr{m}$ (where α is positive). This means that the algebra $\mathscr{g}$ is "symmetric", i.e.

$$[\mathscr{k}, \mathscr{k}] \subset \mathscr{k}, \quad [\mathscr{k}, \mathscr{m}] \subset \mathscr{m}, \quad [\mathscr{m}, \mathscr{m}] \subset \mathscr{k}. \tag{1.3}$$

The curvature tensor R has components in $\mathscr{m}$ defined by

$$e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]]. \tag{1.4}$$

Equation (1.1) can be written as a zero-curvature condition

$$\partial_x A_t - \partial_t A_x + [A_x, A_t] = 0, \tag{1.5}$$

where

$$A_x = \lambda E + A_x^0, \tag{1.6a}$$

$$A_t = \lambda^2 E + \lambda A_x^0 + [E, \partial_x A_x^0] + 1/2[A_x^0[A_x^0, E]], \tag{1.6b}$$