Commun. Math. Phys. 111, 161–166 (1987)

Communications in
Mathematical
Physics

On the Second Eigenfunctions of the Laplacian in $\mathbf{R}^2$*

Chang-Shou Lin**

Centre for Mathematical Analysis, Australian National University, GPO Box 4, Canberra ACT 2601, Australia

Abstract. A conjecture about the nodal line of a second eigenfunction states that the nodal line of a second eigenfunction divides the domain Ω by intersecting with the boundary of Ω transversely, where Ω is a bounded convex domain of $\mathbf{R}^2$. We prove this conjecture provided Ω has a symmetry. Also, we prove the multiplicity of the second eigenvalue is two at most provided Ω is a bounded convex domain of $\mathbf{R}^2$.

1. Introduction

An eigenfunction φ is meant to be a solution of Dirichlet's problem:

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{n=1}^{n} (\partial^2/\partial x_i^2)$ is the Laplacian, Ω is a bounded smooth domain in $\mathbb{R}^n$, and λ is a constant (i.e. the corresponding eigenvalue). It is well known that the first eigenfunction is positive in Ω, and all higher eigenfunctions must change sign. The nodal set of an eigenfunction φ is defined to be the closure of $\{x\in\Omega \mid \varphi(x) = 0\}$. The Courant nodal domain theorem [2] tells us that the nodal set of a kth eigenfunction divides the domain Ω into at most k subregions. We do not know the topology of the nodal set in general, even for the simplest case $n = 2$. A conjecture about the nodal line (i.e. $n = 2$) of a second eigenfunction states that:

(*) the nodal line of a second eigenfunction divides the domain Ω by intersecting its boundary at exactly two points if Ω is convex. (See [5, 6]).

Throughout the paper, Ω is always assumed a bounded *smooth convex* domain in $\mathbf{R}^2$. L. Payne [5] proved the conjecture provided the domain Ω is symmetric with respect to one line. In this paper, we will prove (*) holds true if Ω is symmetric under a rotation with angle $2\pi p/q$, where p, q are positive integers. As a corollary of (*), we

---

* Supported in part by NSF DMS 84-09447

** Home Institution: Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA