Commun. Math. Phys. 111, 101–121 (1987)

Communications in Mathematical Physics

Renormalization Group in the Local Potential Approximation

Giovanni Felder

Institut des Hautes Etudes Scientifiques, 35, route de Chartres, F-91440 Bures-sur-Yvette, France

Abstract. In the local potential approximation, renormalization group equations reduce to a semilinear parabolic partial differential equation. We derive this equation and show the relation with the hierarchical model. We construct a family of non-trivial fixed points u^*_{2n}, $n=2,3,4,\ldots$, which have the form of n-well potentials and exist in the ranges of dimensions $2<d<d_n=2+2/(n-1)$. As $d\uparrow d_n$, u^*_{2n} tends to zero. For the Wilson fixed point u^*_4, we give bounds on critical exponents. In the case of dipole gas in this approximation we show that no non-trivial fixed points exist.

1. Introduction

Non-trivial fixed points of the renormalization group (RG) play a crucial role in the understanding of statistical mechanics systems in the vicinity of the critical point [1]. In the case of a symmetric scalar field (a classical statistical mechanics system with one-component order parameter) the non-trivial fixed points are expected to appear as bifurcating from the trivial massless fixed point as one varies continuously the dimension d of space [2]. These bifurcations occur at the thresholds $d_n=2+2/(n-1)$, $n=2,3,4,\ldots$, where the linearized RG acquires a zero mode (see Fig. 1. The dotted lines represent branches which are believed to be unphysical). The fixed point relevant for three dimensional physics is given by extrapolating to $d=3$ the branch bifurcating at $4=d_2$ dimensions.

This pattern is not well understood from a rigorous point of view, but some pieces of it were established in toy models like Dyson's hierarchical one [3]: Bleher and Sinai [4] proved for this model the existence of a non-trivial fixed point if $d=d_n-\varepsilon$, where $\varepsilon>0$ is small enough. Their result was refined by Collet and Eckmann who proved that the ε-expansion is asymptotic [5]. Gawedzki and Kupiainen considered the case of an N-component spin system in the hierarchical approximation and constructed a fixed point in three dimensions for N large enough [6]. Recently Witter and Koch [7] succeeded in constructing a fixed point