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Some Inequalities for Norm Ideals

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Abstract. Several inequalities for norms of operators are extended to more operators and/or to more norms. These include results of Halmos and Bouldin on approximating a normal operator by another with restricted spectrum, the Powers–Størmer and the van Hemmen–Ando inequalities for the distance between the square roots of two positive operators and also some recent generalisations of these latter results by Kittaneh.

1. Introduction

In this note we obtain extensions of some inequalities for Hilbert space operators which are of importance in some problems of quantum physics and quantum chemistry.

The first problem we consider is called the *spectral approximation problem* for normal operators. Let K be a given closed subset of the complex plane $\mathbb{C}$, and let $\mathcal{N}(K)$ denote the set of all normal operators with spectrum contained in K. Given any normal operator A what element of $\mathcal{N}(K)$ is nearest to A? Halmos [8] showed that if F is any Borel measurable distance minimising retract onto K (i.e. a Borel measurable map from $\mathbb{C}$ onto K which satisfies the inequality $|z - F(z)| \leqq |z - w|$ for all z in $\mathbb{C}$ and w in K) then $\|A - F(A)\| \leqq \|A - N\|$ for every $N \in \mathcal{N}(K)$. This result was extended by Bouldin [3] who showed that for all Schatten p-norms, $p \geqq 2$, $\|A - F(A)\|_p \leqq \|A - N\|_p$. (This statement is to be interpreted to mean that if there exists an N in $\mathcal{N}(K)$ such that $A - N$ is in the Schatten class C_p for some $p \geqq 2$, then $A - F(A)$ also belongs to this class and the above inequality holds.) We refer the reader to the bibliography in [3] for the connection between this problem and some problems arising in molecular orbital calculations in quantum chemistry.

The question for the cases $1 \leqq p < 2$ has been left unanswered by Bouldin. We give below an example to show that the Halmos–Bouldin inequality does not extend to these cases. We then show that there is an interesting class of norms, which includes the p-norms for $p \geqq 2$, to which this inequality can be extended. Further, we show that if the set K is convex then this inequality holds for all unitarily invariant norms.

After this, we consider the inequality of Powers and Størmer [14], derived in the course of their work on free states of the canonical anti-commutation relations. They