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Parallel Transport in the Determinant Line Bundle: The Non-Zero Index Case

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Abstract. For a product family of Weyl operators of possibly non-zero index on a compact manifold X, we express parallel transport in the determinant line bundle in terms of the spectral asymmetry of a Dirac operator on $\mathbb{R} \times X$. This generalizes the results of [7], where we dealt only with invertible operators.

0. Introduction

Let X be a compact spin manifold of even dimension with spin bundle $S = S_+ \oplus S_- \to X$ and let $E \to X$ be a hermitian vector bundle over X. Let $\bar{S}$ and $\bar{E}$ be the pullbacks of S and E to $\mathbb{R} \times X$ with the induced inner products and let $\bar{\nabla}^E$ be a connection on $\bar{E}$. Thus $\bar{\nabla}^E = d_{\mathbb{R}} + \theta + \nabla^E_{(\cdot)}$, where $\theta \in \Omega^1(\mathbb{R}) \otimes C^\infty(X, \operatorname{End} E)$ and for each $y \in \mathbb{R}$, ∇^E_y is a connection on $E \to X$. Let ∂_y be the Weyl operators $\partial_y : L^2(X, S_+ \otimes E) \mapsto L^2(X, S_- \otimes E)$ coupled to the connection ∇^E_y and the (y-independent) metric on X.

The constructions of [5] applied to these data yield a smooth determinant line bundle $\mathscr{L}$ over $\mathbb{R}$ with a natural hermitian metric and compatible connection. If index $\partial_y = 0$, $\mathscr{L}$ has a canonical section. In [8] we assumed that for all y, $\operatorname{Ker} \partial_y = 0$ and $\operatorname{Ker} \partial_y^\dagger = 0$, and we gave a formula expressing parallel transport in $\mathscr{L}$ in terms of this section and the spectral asymmetry $\eta(H)$ of the formally self-adjoint Dirac operator H on $L^2(\mathbb{R} \times X, \bar{S} \otimes \bar{E})$ coupled to the connection $\bar{\nabla}^E$ and the product metric on $\mathbb{R} \times X$.

In this paper we investigate parallel transport in the case that index ∂_y is not necessarily zero. We continue to assume that $\operatorname{Ker} \partial_y = 0$, but now weaken the assumption $\operatorname{Ker} \partial_y^\dagger = 0$ by assuming only that there exists a $V_- \subset L^2(X, S_- \otimes E)$ which is a complement to $\operatorname{Ker} \partial_y^\dagger$ for all y. Let $V_-^\perp$ be the orthogonal complement of V_- viewed as a trivial sub-bundle of the Hilbert bundle $\mathscr{H}_- = \mathbb{R} \times L^2(X, S_- \times E)$, and give V_- the connection induced by orthogonal projection

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