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Limit Sets of S-Unimodal Maps with Zero Entropy

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Abstract. One-dimensional mappings "at the limit of period doubling" are studied in this paper without the use of the renormalization theory of Feigenbaum and others. The principal result is that the attracting part of the nonwandering set is a Cantor set of measure zero under the additional assumption that the map has negative Schwarzian derivative.

The topological structure of unimodal maps of the interval with negative Schwarzian derivative has been completely characterized [2]. The measure theoretic properties of these maps are less thoroughly understood. Here we study maps in one particular topological equivalence class, namely those topologically equivalent to the "Feigenbaum fixed point" [1]. These maps lie at the accumulation of period doubling bifurcations in one parameter families. Without appeal to the properties of the fixed point function or renormalization arguments, we give an elementary geometric proof that the limit sets of these mappings are Cantor sets of Hausdorff dimension smaller than one. In particular, the limit sets of all trajectories have Lebesgue measure zero and the mappings do not support absolutely continuous invariant measures.

Theorem. *Let $f:R\to R$ be an even C^3 map with*

(1) *a single critical point* 0 *which is a nondegenerate maximum,*

(2) *negative Schwarzian derivative $Sf=(f'''/f')-(3/2)(f''/f')^2$, and*

(3) *nonwandering set the union of two unstable fixed points, one periodic orbit of period 2^n for each $n>0$, and a Cantor set Λ without periodic points.*

Then Λ has Lebesgue measure zero.

Remark. The theorem is not the most general which can be proved. In particular, one can allow degenerate critical points which are not flat and relax the assumption that f is even. The proof is presented here in only the simplest case for the sake of clarity.

The proof of the theorem proceeds in several steps which we isolate as separate statements. First we recall some of the topological properties of an f satisfying the