Commun. Math. Phys. 110, 625–640 (1987)

Communications in
Mathematical
Physics

Global Stability of a Class of Discontinuous Dual Billiards

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Abstract. An infinite-parameter family of discontinuous area-preserving maps is studied, using geometrical methods. Necessary and sufficient conditions are determined for the existence of some bounding invariant sets, which guarantee global stability. It is shown that under some additional constraints, all orbits become periodic, most of them Lyapounov stable, and with a maximal period in any bounded domain of phase space. This yields a class of maps acting on a discrete phase space.

1. Introduction

In this work we shall be concerned with the stability problem of an infinite-parameter family of piecewise linear discontinuous area-preserving maps. At the same time, we intend to exploit the peculiar properties of this class of systems to reproduce in a simplified form some features of smooth nonintegrable transformations.

A generic smooth canonical map of the plane is known to possess both regular and irregular orbits. Regular orbits correspond to irrational rotations on invariant circles, due to the KAM theorem [1], whereas the irregular ones, which develop in the place of rational rotations, are still not fully understood. It is known, however, that about transverse homoclinic intersections of separatrices, hyperbolic invariant sets exist over which the map reduces to a shift [1]. The truly random evolution which characterizes this class of solutions has placed them beyond the scope of modern analytical methods.

The main difficulty to be overcome is rooted in the very definition of dynamical instability, which is essentially equivalent to asymptotic ($|t| \to \infty$) discontinuous dependence of the solutions upon the initial conditions [2]. This property, shared not only by all unstable periodic orbits, but rather by the entire set of intersecting separatrices, is the source of that infinite variety of topologically distinct solutions which is a synonym for random dynamics. An additional complication stems from the fact that in smooth nonintegrable systems regular and irregular motions always form a nested pattern repeating itself on every scale.