Commun. Math. Phys. 110, 617–623 (1987)

Communications in
Mathematical
Physics


# Long Time Behaviour of an Infinite Particle System

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**Abstract.** The long time behaviour of the semi-infinite Toda lattice is deduced from a set of identities for the squared eigenfunctions of the Toda flow.

## 1. Introduction

Let $\ell_2^+$ be the real Hilbert space of sequences $u=(u_1, u_2, \ldots)$, $u_i \in \mathbb{R}$ such that $\|u\|^2 = \sum_{i=1}^{\infty} u_i^2 < \infty$, and let $D=\{u \in \ell_2^+ : u_i \neq 0$ for finitely many $i\}$. Recently, the study of the semi-infinite Toda lattice [3] has led us to consider isospectral flows on bounded symmetric operators $L$ in $\ell_2^+$. More precisely, if $L_0$ is a bounded self-adjoint operator on $\ell_2^+$, then, by extending the well-known $QR$ decomposition for matrices to the present case, one has

$$e^{tL_0} = Q(t)\,R(t), \qquad t \in \mathbb{R},$$

where $Q(t)$ is orthogonal $(Q^T(t)\,Q(t) = Q(t)\,Q^T(t) = I)$ and $R(t)$ is upper triangular $[R_{ij}(t)=0$ for $i>j]$ with $R_{ii}(t)>0$. Clearly, the map $L_0 \to \phi(t, L_0) \equiv L(t) \equiv Q^T(t)\,L_0 Q(t)$ defines an isospectral flow on bounded symmetric operators on $\ell_2^+$, and we shall refer to it as the *Toda flow*.

In [1], we develop new techniques to study various properties of the isospectral flows which are related to the Toda flow. In particular, we have a dynamical version of the min-max theorem from which we derive the following

**Theorem.** *If $L_0$ is a Jacobi operator*[1]*, then for all $t$, $\phi(t, L_0)$ is also a Jacobi operator. Moreover, as $t \to \infty$, $\phi(t, L_0)$ converges strongly to a diagonal operator $A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots)$, where*

$$\alpha_i = \inf_{u_1, \ldots, u_{i-1}} \sup_{\substack{u \perp \{u_1, \ldots, u_{i-1}\} \\ \|u\|=1}} (u, L_0 u).$$

[1] A Jacobi operator is a real, bounded, symmetric, tridiagonal operator in $\ell_2^+$ with strictly positive off-diagonal entries