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# A Block Spin Construction of Ondelettes[1]. Part I: Lemarié Functions

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**Abstract.** Using block spin assignments, we construct an $L^2$-orthonormal basis consisting of dyadic scalings and translates of just a finite number of functions. These functions have exponential localization, and for even values of a construction parameter $M$ one can make them class $C^{M-1}$ with vanishing moments up to order $M$ inclusive. Such a basis has an important application to phase cell cluster expansions in quantum field theory.

## 1. Introduction

Quite recently Y. Meyer et al. [1, 2, 3] have constructed very useful bases of ondelettes (wavelets) to solve certain problems in functional analysis. These new functions are now expected to have applications to several areas of physics. They have already had an impact on constructive quantum field theory [4, 5, 6].

A basis of ondelettes is defined to be an orthonormal basis—say for $L^2(\mathbb{R}^d)$—whose functions are dyadic scalings (from $2^{-\infty}$ to $2^{\infty}$) and translates of just a finite number of them. The most familiar example is the standard basis of Haar functions on $\mathbb{R}^d$. Indeed, Battle and Federbush [4, 5] used a polynomial generalization of the Haar basis to develop a phase cell cluster expansion a few years ago. This basis has the following useful properties:

(a) The basis consists of all dyadic scalings and translates of a finite collection $\psi_1, \ldots, \psi_n$ of functions.

(b) $\psi_i$ is a piecewise polynomial supported on the cube associated with it. Thus we have sharp localization but poor regularity.

(c) For all multi-indices $\alpha$ for which $|\alpha|$ is less than or equal to a certain construction parameter,

$$\int \psi_i(x)\, x^{\alpha}\, dx = 0. \tag{1.1}$$

Equivalently, $\hat{\psi}_i(p)$ vanishes to some finite order at $p = 0$.

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