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The Index of Scattering Operators of Dirac Equations

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Abstract. A new index formula of Atiyah Singer type for scattering operators is proved. The index corresponds to the vacuum polarization of the Fermion (on the Minkowski space) coupled to an external non abelian gauge field.

1 Introduction

Geometric features of gauge theories have been extensively investigated since the discovery of instantons. In spite of several successes, the methods have been based on the compactification of the space-time manifold, which is of great disadvantage if we wish to find precise relations between the obtained results and realistic quantum field theories. Even though the non-compactness of $\mathbb{R}^4$ is controlled by the boundary condition at infinity, we don't know how to relate results for compact Riemannian manifolds to the Minkowski space (the theory of elliptic operators, e.g. Atiyah-Singer's index theorem to the theory of hyperbolic differential operators). In the Minkowski space, we have no effective geometric tools for the study of anomalies and other topological effects.

The motivation of the present work is to find a geometric invariant of gauge theories in quantum systems on the Minkowski space. We explain the background of our results in more detail now.

Consider a (second) quantized *charged* Fermion coupled with an external field on a Fock space. Mathematically the Fermion field is an element of a CAR (canonical anticommunication relations) algebra which is isomorphic to an infinite dimensional Clifford algebra $\mathfrak{A}$ on a complex Hilbert space $\mathscr{H}$ with an antiunitary involution. The Fock representation is an irreducible representation of $\mathfrak{A}$ with a special vector Ω called vacuum. This representation is completely specified by a projection P on $\mathscr{H}$. See [1].

In most physical situations, the representation $u(g)$ on $\mathscr{H}$ of a compact group G is given which is canonically lifted to an action α_g of G on $\mathfrak{A}$ via Bogoliubov automorphisms. α_g is identified with the global gauge transformation, and the fixed point subalgebra $\mathfrak{A}^G$ of this action is regarded as the observable algebra.

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