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Time Decay of Solutions to the Cauchy Problem for Time-Dependent Schrödinger–Hartree Equations

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Abstract. We consider the time-dependent Schrödinger–Hartree equation

$$iu_t + \Delta u = \left(\frac{1}{r} * |u|^2\right) u + \lambda \frac{u}{r}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1}$$

$$u(0,x) = \phi(x) \in \Sigma^{2,2}, \quad x \in \mathbb{R}^3, \tag{2}$$

where $\lambda \geqq 0$ and $\Sigma^{2,2} = \{g \in L^2; \ \|g\|^2_{\Sigma^{2,2}} = \sum_{|\alpha| \leqq 2} \|D^\alpha g\|_2^2 + \sum_{|\beta| \leqq 2} \|x^\beta g\|_2^2 < \infty\}$.

We show that there exists a unique global solution u of (1) and (2) such that

$$u \in C(\mathbb{R}; H^{1,2}) \cap L^\infty(\mathbb{R}; H^{2,2}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}; \Sigma^{2,2})$$

with

$$u_t \in L^\infty(\mathbb{R}; L^2).$$

Furthermore, we show that u has the following estimates:

$$\|u(t)\|_{2,2} \leqq C, \quad \text{a.e.} \quad t \in \mathbb{R},$$

and

$$\|u(t)\|_\infty \leqq C(1+|t|)^{-1/2}, \quad \text{a.e.} \quad t \in \mathbb{R}.$$

1. Introduction and Main Results

We consider the time decay of solutions to the Cauchy problem for the equation in $L^2 = L^2(\mathbb{R}^3)$

$$iu_t + \Delta u = f(|u|^2)u + \lambda V u, \quad t \in \mathbb{R}, \tag{1.1}$$

$$u(0) = \phi, \tag{1.2}$$

where $u_t = \partial_t u$, $f(|u|^2) = |x|^{-1} * |u|^2 = \int_{\mathbb{R}^3} |u(t,y)|^2/|x-y| dy$, $\lambda \geqq 0$, $V = 1/|x|$ and ϕ