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Spectral Functions, Special Functions and the Selberg Zeta Function

A. Voros*

Service de Physique Théorique, CEN-Saclay, F-91191 Gif-sur-Yvette Cedex, France

Abstract. The functional determinant of an eigenvalue sequence, as defined by zeta regularization, can be simply evaluated by quadratures. We apply this procedure to the Selberg trace formula for a compact Riemann surface to find a factorization of the Selberg zeta function into two functional determinants, respectively related to the Laplacian on the compact surface itself, and on the sphere. We also apply our formalism to various explicit eigenvalue sequences, reproducing in a simpler way classical results about the gamma function and the Barnes G-function. Concerning the latter, our method explains its connection to the Selberg zeta function and evaluates the related Glaisher–Kinkelin constant A.

Introduction

This note studies from a general and systematic point of view certain spectral functions, especially determinants and generalized zeta functions, which can be associated with a numerical sequence $\{\lambda_k\}$ (typically the spectrum of a certain differential operator). In particular we reduce to elementary manipulations the evaluation of certain functional determinants which find applications in high energy physics (string theory, see [1] and references therein) and in differential geometry (analytic torsion, see [2, 3] and references therein). Our main result (Sect. 7) is actually an explicit factorization of the Selberg zeta function into two functional determinants, one of which is expressible in terms of the Barnes G-function; some recently published formulae involving the Selberg zeta function [1, 3] follow from ours by specializing the value of the spectral variable.

Our formalism itself is developed in Sect. 2 to 4; its connection to asymptotic (semi-classical) expansions is demonstrated in Sect. 5, and Sect. 6 is devoted to examples drawn from number theory (the Euler gamma function and the Barnes G-function) and from quantum mechanics (homogeneous Schrödinger operators like the quartic oscillator).

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* Member of CNRS