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Existence of Solutions for Schrödinger Evolution Equations

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Abstract. We study the existence, uniqueness and regularity of the solution of the initial value problem for the time dependent Schrödinger equation $i\partial u/\partial t = (-1/2)\Delta u + V(t,x)u$, $u(0) = u_0$. We provide sufficient conditions on $V(t,x)$ such that the equation generates a unique unitary propagator $U(t,s)$ and such that $U(t,s)u_0 \in C^1(\mathbb{R}, L^2) \cap C^0(\mathbb{R}, H^2(\mathbb{R}^n))$ for $u_0 \in H^2(\mathbb{R}^n)$. The conditions are general enough to accommodate moving singularities of type $|x|^{-2+\varepsilon}(n \geqq 4)$ or $|x|^{-n/2+\varepsilon}(n \leqq 3)$.

1. Introduction, Assumptions and Theorems

In this paper, we study the existence, uniqueness and regularity of the solution of the initial value problem for the time dependent Schrödinger equation in $\mathbb{R}^n$:

$$\begin{aligned} &i\partial u/\partial t = -(1/2)\Delta u + V(t,x)u, \quad t \in [-T, T] = I_T, \quad x \in \mathbb{R}^n, \\ &u(s,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ and $V(t,x)$ is a real valued function. We regard Eq. (1.1) as an evolution equation in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$:

$$idu/dt = H(t)u, \quad H(t) = -(1/2)\Delta + V(t,x), \quad u(s) = u_0, \tag{1.2}$$

and treat the problem by using the perturbation technique and the well-known $L^p - L^q$-type estimates for the free propagator $\exp(it\Delta/2)$. We shall give sufficient conditions on $V(t,x)$ such that Eq. (1.2) uniquely generates a strongly continuous unitary propagator $\{U(t,s)\}$ on $\mathscr{H}$, and such that $U(t,s)u_0 \in C(I_T, H^2(\mathbb{R}^n)) \cap C^1(I_T, \mathscr{H})$ for every $u_0 \in H^2(\mathbb{R}^n)$. The conditions are general enough to accommodate potentials which have moving singularities of type $|x|^{-2+\varepsilon}$ for $n \geqq 4$ and $|x|^{-n/2+\varepsilon}$ for $n \leqq 3$, $\varepsilon > 0$.

We consider, along with Eq. (1.2), the integral equation

$$u(t) = U_0(t-s)u_0 - i\int_s^t U_0(t-\tau)V(\tau)u(\tau)d\tau, \tag{1.3}$$

where $U_0(t) = \exp(it\Delta/2)$ and $V(t)$ is the multiplication operator by $V(t,x)$. For an interval I and m, $\rho \geqq 1$, $L^{m,\rho}(I)$ is the Banach space of $L^m(\mathbb{R}^n)$-valued ρ-summable