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The Goryachev-Chaplygin Top and the Toda Lattice

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Abstract. The Goryachev-Chaplygin top is a rigid body rotating about a fixed point with principal moments of inertia A, B, C satisfying $A=B=4C$ and with center of mass lying in the equatorial plane. The problem is algebraically completely integrable as a linear flow on a hyperelliptic Jacobian, only upon putting the principal angular momentum in the horizontal plane.

This system admits asymptotic solutions with fractional powers in t and depending on 4 degrees of freedom. As a consequence, the affine invariant surfaces of the Chaplygin top are double covers of the hyperelliptic Jacobian above, ramified along two translates of the theta divisor, touching in one point. This system is an instance of a (master) system of differential equations in 7 unknowns having 5 quadratic constants of motion; a careful analysis of this system reveals an intimate (rational) relationship with the 3-body periodic Toda lattice.

The Goryachev-Chaplygin top is a rigid body rotating about a fixed point with principal moments of inertia A, B, C satisfying $A=B=4C$ and with center of mass lying in the equatorial plane (through the fixed point) going with the moments A and B. First introduced by Goryachev [7] in 1900 and later integrated by Chaplygin [5] in terms of hyperelliptic integrals, we learned about the system from Golubev [6]. This motion has an extra-constant of motion, only upon putting the principal angular momentum in the horizontal plane. After some rescaling, the equations of motion take on the form:

$$\begin{aligned} \dot{x} &= 3yz, & \dot{u} &= 4zv - yw, \\ \dot{y} &= -3xz - 4w, & \dot{v} &= xw - 4zu, \\ \dot{z} &= 4v, & \dot{w} &= yu - xv. \end{aligned} \tag{1}$$

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