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Stability of Classical Solutions of Two-Dimensional Grassmannian Models

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Abstract. We show that the only finite-action solutions of the two-dimensional Grassmannian σ-model that are stable under small fluctuations are the (anti-)instanton solutions.

0. Introduction

(0.1) The two-dimensional Grassmannian σ-model is a field theory which shares many of the properties of the (more complicated) four-dimensional non-abelian gauge theories: for instance, the action is conformally invariant, there is a topological charge and the associated (anti-)instantons minimise the action among all fields with the same charge. For a survey of this theory, see [11].

(0.2) It is of interest to know whether there exist any non-instanton solutions in this model that are stable under small fluctuations. It is the purpose of this article to answer this question in the negative; thus all non-(anti-)instanton solutions are saddle points for the action. Our technique uses methods of Algebraic Geometry to ensure a sufficiently large number of non-positive modes for the fluctuation operator so that stability is only possible for (anti-)instanton solutions. These non-positive modes are essentially provided by solutions of the background fermion problem.

1. Preliminaries

(1.1) The non-linear σ-model is a field theory where the dynamical variable takes values in a Riemannian manifold (N, h). The Lagrangian density and action for this model are given by

$$L(\varphi)=h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial_\mu\varphi^\beta\,, \qquad S=\int L d^n x\,. \tag{1}$$

We are interested in finite-action solutions of the equations of motion, which are known to mathematicians as *harmonic maps* (see e.g. [3]). We shall restrict attention to the 2-dimensional Euclidean version of the model, which is of most interest to physicists since it shares a number of properties with $4d$ non-abelian gauge theories. In particular, in this case, the action is conformally invariant and, by a result of Sacks and Uhlenbeck [9], any finite-action solution of the equations of motion extends to a solution on the conformal compactification of $\mathbf{R}^2$, the Riemann sphere $S^2=\mathbf{R}^2\cup\{\infty\}$. Henceforth therefore, we shall suppose, without